

# ON THE NEGATIVE $K$ -THEORY OF SCHEMES IN FINITE CHARACTERISTIC

AMALENDU KRISHNA

**ABSTRACT.** We show that if  $X$  is a  $d$ -dimensional scheme of finite type over a perfect field  $k$  of characteristic  $p > 0$ , then  $K_i(X) = 0$  and  $X$  is  $K_i$ -regular for  $i < -d - 2$  whenever the resolution of singularities holds over  $k$ . This proves the  $K$ -dimension conjecture of Weibel [22, 2.9] (except for  $-d - 1 \leq i \leq -d - 2$ ) in all characteristics, assuming the resolution of singularities.

## 1. INTRODUCTION

It is by now well known that the negative  $K$ -theory of singular schemes is non-zero in general and bears a significant information about the nature of the singularity of  $X$ . Hence it is a very interesting question to know how much of the negative  $K$ -theory of a singular variety can survive. A beautiful answer was given in terms of the following very general conjecture of Weibel.

**Conjecture 1.1** (Weibel, [22]). *Let  $X$  be a Noetherian scheme of dimension  $d$ . Then  $K_i(X) = 0$  for  $i < -d$  and  $X$  is  $K_{-d}$ -regular.*

This conjecture was proved recently by Cortinas-Haesemeyer-Schlichting-Weibel [4, Theorem 6.2] for schemes of finite type over a field of characteristic zero. If  $X$  is a scheme of finite type over a field of positive characteristic, the above conjecture was proved by Weibel [23] provided the dimension of  $X$  is at most two. Our aim in this paper is to prove the conjecture for a  $d$ -dimensional scheme in the positive characteristic (except for  $i = -d - 1$ ) assuming the resolution of singularities. A *variety*  $X$  in this paper will mean a scheme of finite type over an infinite perfect field  $k$  of characteristic  $p > 0$ , which will be fixed throughout this paper.

**Definition 1.2** (Resolution of singularities). We will say that the resolution of singularities holds over  $k$  if given any equidimensional scheme  $X$  of finite type over  $k$ , there exists a sequence of monoidal transformations

$$X_r \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$$

such that the following hold:

- (i) the reduced subscheme  $X_r^{\text{red}}$  is smooth over  $k$ ,
- (ii) the center  $D_i$  of the monoidal transformation  $X_{i+1} \rightarrow X_i$  is smooth and connected and nowhere dense in  $X_i$ .

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The resolution of singularities holds over fields of characteristic zero by the work of Hironaka [10, Theorem 1\*]. For fields of positive characteristics, this problem is still not known though widely expected to be true. Recently, Hironaka [11, ] has outlined a complete program to solve this problem and work on this program is in progress.

Recall that a variety  $X$  is said to be  $K_i$ -regular if the natural map  $K_i(X) \rightarrow K_i(X[T_1, \dots, T_r])$  is an isomorphism for all  $r \geq 1$ , where  $K_i(X)$  is the  $i$ th stable homotopy group of the non-connective spectrum  $K(X)$  of perfect complexes on  $X$ . It is known from a result of Vorst [21] that a scheme which is  $K_i$ -regular, is also  $K_j$ -regular for  $j \leq i$ . We now state the main result of this paper.

**Theorem 1.3.** *Assume that the resolution of singularities holds over  $k$  and let  $X$  be a variety of dimension  $d$  over  $k$ . Then*

- (i)  $K_i(X, \mathbb{Z}/n) = 0$  for  $i < -d - 1$  and for all  $n \geq 1$ .
- (ii)  $K_{-d-2}(X[T_1, \dots, T_j])$  is a divisible torsion group for all  $j \geq 0$ .
- (iii)  $K_i(X) = 0$  and  $X$  is  $K_i$ -regular for  $i < -d - 2$ .

## 2. $K$ -THEORY AND TOPOLOGICAL CYCLIC HOMOLOGY

In this section, we briefly recall the topological cyclic homology of rings and schemes and the cyclotomic trace map from the  $K$ -theory to the topological cyclic homology. We then show that the homotopy fiber of this trace map satisfies the descent for the  $cdh$ -topology.

Let  $p$  be a fixed prime number. Let  $A$  be a commutative ring which is essentially of finite type over a field  $k$  of characteristic  $p$ . Recall from [5] that the topological Hochschild spectrum  $T(A)$  is a symmetric  $\mathbb{S}$ -spectrum, where  $\mathbb{S}$  is the circle group. Let  $C_r \subset \mathbb{S}$  be the cyclic subgroup of order  $r$ . Then one defines

$$TR^n(A; p) = F\left(\mathbb{S}/C^{p^{n-1}}, T(A)\right)^{\mathbb{S}}$$

to be the fixed point spectrum of the function spectrum  $F\left(\mathbb{S}/C^{p^{n-1}}, T(A)\right)$ .

There are the Frobenius and the restriction maps of spectra

$$F, R : TR^n(A; p) \rightarrow TR^{n-1}(A; p).$$

The spectrum  $TC^n$  is defined as the homotopy equalizer of the maps  $F$  and  $R$ , i. e.,

$$TC^n(A; p) = eq\left(TR^n(A; p) \xrightarrow[F]{R} TR^n(A; p)\right),$$

and the topological cyclic homology spectrum  $TC(A; p)$  is defined as the homotopy limit

$$TC(A; p) = \text{holim} TC^n(A; p).$$

One similarly defines

$$TR(A; p) = \text{holim}_R TR^n(A; p)$$

$$TF(A; p) = \text{holim}_F TR^n(A; p).$$

It was shown by Geisser and Hesselholt in *loc. cit.* that the topological Hochschild and cyclic homology satisfy descent for a Cartesian diagram of rings and they were able to define these homology spectra for a Noetherian scheme  $X$  using the Thomason's construction of the hypercohomology spectrum [17, 1.33]. Recently, Blumberg and Mandell [1] have made a significant progress in the study of the topological Hochschild and cyclic homology of schemes. They globally define the spectra  $T(X)$  and  $TC(X)$  for a Noetherian scheme  $X$  as the topological Hochschild and cyclic homology spectra of the spectral category  $D(Perf/X)$  which is the Thomason's derived category of perfect complexes on  $X$  [18]. They show that their definition of these spectra coincides with the above definition for affine schemes. They also prove the localization and the Zariski descent properties of the topological Hochschild and cyclic homology of schemes. We refer to [1] for more details. In this paper, the topological Hochschild and cyclic homology of schemes will be considered in the sense of [1]. For any symmetric spectrum  $E$  and for  $n \geq 1$ , let  $E/p^n$  denote the smash product of  $E$  with a mod  $p^n$  Moore spectrum  $\Sigma^\infty/p^n$ .

Let  $K(X)$  denote the Thomason's non-connective spectrum of the perfect complexes on  $X$ . For a ring  $A$ , there is a cyclotomic trace map [2] of non-connective spectra

$$K/p^n(A) \xrightarrow{tr} TC/p^n(A; p).$$

Since  $K$ -theory satisfies Zariski descent by [18] and so does the topological cyclic homology by [1], taking the induced map on the Zariski hypercohomology spectra gives for any Noetherian scheme  $X$ , the cyclotomic trace map of spectra

$$(2.1) \quad K/p^n(X) \xrightarrow{tr} TC/p^n(X; p).$$

Let  $L^n(X)$  denote the homotopy fiber of the trace map in 2.1. If  $Sch/k$  denotes the category of varieties over  $k$ , then one gets a presheaf of homotopy fibrations of spectra on  $Sch/k$

$$(2.2) \quad L^n \rightarrow K/p^n \xrightarrow{tr} TC/p^n(-; p).$$

### 3. CDH-DESCENT FOR $L^n$

We recall from [4] that a presheaf of spectra  $\mathcal{E}$  on the category  $Sch/k$  satisfies the *Mayer-Vietoris property* for a Cartesian square of schemes

$$(3.1) \quad \begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

if applying  $\mathcal{E}$  to this square results in a homotopy Cartesian square of spectra. We say that  $\mathcal{E}$  satisfies the Mayer-Vietoris property for a class of squares provided it satisfies this property for each square in that class. One says that the presheaf of spectra  $\mathcal{E}$  is *invariant under infinitesimal extension* if for any affine scheme  $X$  and a closed subscheme  $Y$  of  $X$  defined

by a sheaf of nilpotent ideals  $\mathcal{I}$ , the spectrum  $\mathcal{E}(X, Y)$  is contractible, where the latter is the homotopy fiber of the map  $\mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ . One says that  $\mathcal{E}$  satisfies the *excision property* if for any morphism of affine schemes  $f : X \rightarrow Y$  and a sheaf of ideals  $\mathcal{I}$  on  $X$  such that  $I \cong f_* f^*(\mathcal{I})$ , the spectrum  $\mathcal{E}(Y, X, \mathcal{I})$  is contractible, where  $\mathcal{E}(X, Y, \mathcal{I})$  is defined as the homotopy fiber of the map  $\mathcal{E}(X, \mathcal{I}) \rightarrow \mathcal{E}(Y, \mathcal{I})$ . An elementary Nisnevich square is a Cartesian square of schemes as above such that  $Y \rightarrow X$  is an open embedding,  $X' \rightarrow X$  is étale and  $(X' - Y') \rightarrow (X - Y)$  is an isomorphism. Then one says that  $\mathcal{E}$  satisfies *Nisnevich descent* if it satisfies the Mayer-Vietoris property for all elementary Nisnevich squares.

We next recall from [20] (see also *loc. cit.*) that a *cd-structure* on a small category  $\mathcal{C}$  is a class  $\mathcal{P}$  of commutative squares in  $\mathcal{C}$  that is closed under isomorphisms. Any such *cd-structure* defines a topology on  $\mathcal{C}$ . We assume in the rest of the paper that our ground field  $k$  admits the resolution of singularities. In this case, the *combined cd-structure* on the category  $Sch/k$  consists of all elementary Nisnevich squares and all abstract blow ups, where an abstract blow-up is a Cartesian square as in 2.2 where  $Y \rightarrow X$  is a closed embedding,  $X' \rightarrow X$  is proper and the induced map  $(X' - Y')_{red} \rightarrow (X - Y)_{red}$  is an isomorphism. The topology generated by the combined *cd-structure* is called the *cdh-topology*. Let  $Sm/k$  denote the category of smooth varieties over  $k$ . Since the resolution of singularities holds over  $k$ , the restriction of the *cd-structure* to the category  $Sm/k$  where abstract blow-ups are replaced by the smooth blow-ups, is complete, bounded and regular (cf. [20, Section 4]). The topology generated by this *cd-structure* on  $Sm/k$  is called the *scdh-topology*. This is just the restriction of the *cdh-topology* on the subcategory  $Sm/k$ . In this paper, we shall consider the local injective model structure on the category of presheaves of spectra on  $Sch/k$  as described in [4].

For the local injective model structure the category of presheaves of spectra on a given topology  $\mathcal{C}$  on  $Sch/k$ , a fibrant replacement of a presheaf of spectra  $\mathcal{E}$  is a trivial cofibration  $\mathcal{E} \rightarrow \mathcal{E}'$  where  $\mathcal{E}'$  is fibrant. We shall write such a fibrant replacement as  $\mathbb{H}_{\mathcal{C}}(-, \mathcal{E})$ . We shall say that  $\mathcal{E}$  satisfies the *cdh-descent* if it satisfies the Mayer-Vietoris property for all elementary Nisnevich squares and all abstract blow-ups. By [4, Theorem 3.4], this is equivalent to the assertion that the map  $\mathcal{E} \rightarrow \mathbb{H}_{cdh}(-, \mathcal{E})$  is a global weak equivalence in the sense that  $\mathcal{E}(X) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{E})$  is a weak equivalence for all  $X \in Sch/k$ .

**Theorem 3.1.** *Let  $\mathcal{E}$  be a presheaf of spectra on  $Sch/k$  such that  $\mathcal{E}$  satisfies excision, is invariant under infinitesimal extension, satisfies Nisnevich descent and satisfies the Mayer-Vietoris property for every blow-up along a regular closed embedding. Then  $\mathcal{E}$  satisfies the *cdh-descent*.*

*Proof.* The proof of this theorem is very similar to the proof of the analogous theorem in *loc. cit.* (Theorem 3.12) when  $k$  has characteristic zero. We only give the brief sketch. As shown above, it suffices to show that the map

$$(3.2) \quad \mathcal{E}(X) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{E})$$

is a weak equivalence for all varieties  $X$  over  $k$ . Since the  $scdh$ -topology on  $Sm/k$  is generated by elementary Nisnevich squares and smooth blow-ups and since the closed embeddings of smooth varieties are regular embeddings, we see that  $\mathcal{E}$  satisfies the  $scdh$ -descent in  $Sm/k$ .

Now assume  $X$  is singular. As explained in *loc. cit.*, the argument goes as in the proof of Theorem 6.4 in [8]. The excision, invariance under infinitesimal extension and Nisnevich descent together imply that  $\mathcal{E}$  satisfies the Mayer-Vietoris property for closed covers and for finite abstract blow-ups. Now if  $X$  is a hypersurface in a smooth scheme, we can follow the proof of Theorem 6.1 in [8] to conclude that 3.2 holds for  $X$  since the resolution of singularities holds over  $k$ , which is also infinite. If  $X$  is a complete intersection inside a smooth  $k$ -scheme, then we can use the hypersurface case, the Mayer-Vietoris for the closed covers and an induction on the embedding dimension of  $X$  to conclude 3.2 for  $X$ . The general case follows from this as shown in [8, Theorem 6.4].  $\square$

**Corollary 3.2.** *The presheaf of spectra  $L^n$  (cf. 2.2) satisfies the  $cdh$ -descent.*

*Proof.* We need to show that  $L^n$  satisfies all the conditions of Theorem 3.1. We have the homotopy fibration of presheaves of spectra

$$L^n \rightarrow K/p^n \rightarrow TC/p^n(-; p).$$

The fact that  $L^n$  satisfies excision was proved by Geisser-Hesselholt [6, Theorem 1]. The invariance of  $L^n$  under infinitesimal extension was proved by McCarthy [15, Main Theorem]. Next we show that  $L^n$  satisfies Nisnevich descent.  $K/p^n$  satisfies Nisnevich descent by [18, Theorem 10.8].  $TC/p^n(-; p)$  satisfies Nisnevich descent by [5, Corollary 3.3.4] and by the agreement of the definition of the topological cyclic homology as given in [5] with that of [1] since the topological cyclic homology of [1] satisfies the Zariski descent (see the discussion in Remark 3.3.5 of [5]). We now consider the following commutative diagram of spectra for a given variety  $X$ .

$$(3.3) \quad \begin{array}{ccccc} L^n(X) & \longrightarrow & K/p^n(X) & \longrightarrow & TC/p^n(X; p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}_{Nis}(X, L^n) & \longrightarrow & \mathbb{H}_{Nis}(X, K/p^n) & \longrightarrow & \mathbb{H}_{Nis}(X, TC/p^n(-; p)) \end{array}$$

Since the top row in the above diagram is a homotopy fibration and the bottom row is a fibrant replacement of the top row, the bottom row is also a homotopy fibration (cf. [17, 1.35], see also [4, Section 5]). Now, since the middle and the right vertical maps are weak equivalences, we see that the left vertical map is also a weak equivalence. This verifies the Nisnevich descent for  $L^n$ . Finally,  $L^n$  satisfies the Mayer-Vietoris property for the blow-up along regular closed embeddings by [1, Theorem 1.4]. We conclude from Theorem 3.1 that  $L^n$  satisfies  $cdh$ -descent.  $\square$

Let  $a$  denote the natural morphism from the  $cdh$ -site to the Zariski site on the category  $Sch/k$ . For any Zariski sheaf  $\mathcal{F}$ , let  $a_{cdh}\mathcal{F}$  denote the  $cdh$ -sheafification of  $\mathcal{F}$ .

**Corollary 3.3.** *For any  $k$ -variety  $X$ , there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H_{cdh}^p(X, a_{cdh}\pi_q(L^n)) \Rightarrow L_{q-p}^n(X),$$

where the differentials of the spectral sequence are  $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q+r-1}$ .

*Proof.* This follows immediately from Corollary 3.2 and the fact that the  $cdh$ -cohomological dimension of  $X$  is bounded by the Krull dimension of  $X$  (cf. [16, Theorem 12.5]).  $\square$

#### 4. VANISHING AND HOMOTOPY INVARIANCE FOR $L^n$

Following [4], we let  $\tilde{C}_j\mathcal{E}$  denote the homotopy cofiber of the natural map  $\mathcal{E} \rightarrow \mathcal{E}(- \times \mathbb{A}^j)$  for any presheaf of spectra  $\mathcal{E}$  on  $Sch/k$ . Note that  $\mathcal{E}(- \times \mathbb{A}^j)$  is a canonical direct sum of  $\mathcal{E}$  and  $\tilde{C}_j\mathcal{E}$  and hence the functor  $\tilde{C}_j$  preserves the homotopy fibration sequences. In particular, we get a presheaf of fibration sequences

$$(4.1) \quad \tilde{C}_j L^n \rightarrow \tilde{C}_j K/p^n \rightarrow \tilde{C}_j TC/p^n(-; p).$$

Furthermore, since  $L^n$  and  $L^n(- \times \mathbb{A}^j)$  satisfy  $cdh$ -descent, we see that  $\tilde{C}_j L^n$  also satisfies  $cdh$ -descent.

**Lemma 4.1.** *For a  $d$ -dimensional variety  $X$ , one has  $L_i^n(X) = 0 = \pi_i \tilde{C}_j L^n(X)$  for all  $j \geq 0$  and  $i < -d - 2$ .*

*Proof.* Using Corollaries 3.2 and 3.3 and [16, Theorem 12.5], it suffices to show that  $a_{cdh}\pi_q(L^n)$  and  $a_{cdh}\pi_q(\tilde{C}_j L^n)$  are zero for  $i < -2$ . The presheaf of fibration sequences 2.1 gives the long exact sequence of presheaves of homotopy groups on  $Sch/k$

$$\cdots \rightarrow L_i^n \rightarrow K/p_i^n \rightarrow TC/p_i^n(-; p) \rightarrow L_{i-1}^n \rightarrow \cdots.$$

Since the sheafification is an exact functor, we get the corresponding long exact sequence of  $cdh$ -sheaves

$$(4.2) \quad \cdots \rightarrow a_{cdh}L_i^n \rightarrow a_{cdh}K/p_i^n \rightarrow a_{cdh}TC/p_i^n(-; p) \rightarrow a_{cdh}L_{i-1}^n \rightarrow \cdots.$$

We similarly get a long exact sequence of  $cdh$ -sheaves

$$(4.3) \quad \begin{aligned} \cdots \rightarrow a_{cdh}\pi_i(\tilde{C}_j L^n) &\rightarrow a_{cdh}\pi_i(\tilde{C}_j K/p^n) \rightarrow a_{cdh}\pi_i(\tilde{C}_j TC/p^n(-; p)) \\ &\rightarrow a_{cdh}\pi_{i-1}(\tilde{C}_j L^n) \rightarrow \cdots. \end{aligned}$$

Since the smooth schemes have no non-zero negative  $K$ -theory, we have  $a_{cdh}K/p^n_i = 0$  for  $i < 0$  and hence there are isomorphisms

$$(4.4) \quad {}_{cdh}TC/p^n_i(-; p) \xrightarrow{\cong} a_{cdh}L^n_{i-1} \quad \text{and} \\ \pi_i \left( \tilde{C}_j TC/p^n(-; p) \right) \xrightarrow{\cong} a_{cdh}\pi_{i-1} \left( \tilde{C}_j L^n \right) \quad \text{for } i < 0.$$

Thus it suffices to show that the left terms of both the isomorphisms vanish for  $i < -1$ . For this, it suffices to show that  $TC_i(A; p, \mathbb{Z}/p^n) = 0$  for  $i < -1$  for any ring  $A$  which is essentially of finite type over  $k$ . One knows from a result of Hesselholt (cf. [9]) that  $TC_i(A; p) = 0$  for  $i < -1$  and the same conclusion then holds with finite coefficients by the exact sequence

$$TC_i(A; p) \rightarrow TC_i(A; p, \mathbb{Z}/p^n) \rightarrow TC_{i-1}(A; p).$$

□

**Lemma 4.2.** *Let  $X$  be a  $k$ -variety of dimension  $d$ . Then there are natural isomorphisms*

$$H_{\mathcal{C}}^d(X, a_{\mathcal{C}}\pi_{-1}(TC/p^n(-; p))) \xrightarrow{\cong} \mathbb{H}_{\mathcal{C}}^{d+1}(X, TC/p^n(-; p)), \\ H_{\mathcal{C}}^d\left(X, a_{\mathcal{C}}\pi_{-1}\left(\tilde{C}_j TC/p^n(-; p)\right)\right) \xrightarrow{\cong} \mathbb{H}_{\mathcal{C}}^{d+1}\left(X, \tilde{C}_j TC/p^n(-; p)\right)$$

for  $\mathcal{C}$  being the Zariski or the  $cdh$ -site.

*Proof.* Since the Zariski or the  $cdh$ -cohomological dimension of  $X$  is bounded by  $d$ , one has a strongly convergent spectral sequence

$$(4.5) \quad E_2^{s,t} = H_{\mathcal{C}}^s(X, a_{Zar}\pi_t(TC/p^n(-; p))) \Rightarrow \mathbb{H}_{\mathcal{C}}^{s-t}(X, TC/p^n(-; p))$$

and the similar spectral sequence holds for the homotopy groups of the  $\tilde{C}_j$  functors.

Since  $TC_i(A; p, \mathbb{Z}/p^n) = 0$  for any ring  $A$  and for any  $i < -1$  as mentioned above, we conclude that for  $s < d$ , one has  $-d - 1 + s < -1$  and hence

$$H_{\mathcal{C}}^s(X, a_{\mathcal{C}}\pi_{-d-1+s}(TC/p^n(-; p))) = 0 = H_{\mathcal{C}}^s\left(X, a_{\mathcal{C}}\pi_{-d-1+s}\left(\tilde{C}_j TC/p^n(-; p)\right)\right)$$

and also  $E_2^{d-2,-2} = 0$ . Hence the above spectral sequence degenerates enough to give the desired isomorphisms. □

**Lemma 4.3.** *Let  $X$  be as in Lemma 4.2. Then the natural maps*

$$H_{cdh}^d(X, a_{cdh}\pi_{-2}(L^n)) \rightarrow \mathbb{H}_{cdh}^{d+2}(X, L^n), \\ H_{cdh}^d\left(X, a_{cdh}\pi_{-2}\left(\tilde{C}_j L^n\right)\right) \rightarrow \mathbb{H}_{cdh}^{d+2}\left(X, \tilde{C}_j L^n\right)$$

are isomorphisms.

*Proof.* Since the smooth schemes have no negative  $K$ -theory, we have  $a_{cdh}K/p^n_i = 0$  for  $i < 0$ . We have also seen above that  $a_{cdh}TC/p^n_i(-; p) = 0$  for  $i < -1$ , and the same vanishing holds for the homotopy groups of the  $\tilde{C}_j$ -functors. We conclude from the exact sequences 4.2 and 4.3 that  $a_{cdh}L^n_i = 0 = a_{cdh}\pi_i(\tilde{C}_jL^n)$  for  $i < -2$ . The spectral sequence 4.5 now implies the lemma.  $\square$

## 5. VANISHING AND HOMOTOPY INVARIANCE FOR $K/p^n$

In this section, we prove the vanishing results for some negative homotopy groups of  $K/p^n$  and  $\tilde{C}_jK/p^n$  using the results of the previous section. We begin with the following result about the sheaves of rings of Witt vectors. For a  $k$ -variety  $X$ , let  $W\mathcal{O}_X$  denote the sheaf of Witt vectors on  $X$  (cf. [12]). The sheaf  $W\mathcal{O}$  is a sheaf of rings on the category  $Sch/k$ .

**Lemma 5.1.** *For any  $k$ -variety of dimension  $d$  and for any  $n \geq 1$ , the natural maps*

$$(5.1) \quad H_{Zar}^d(X, W\mathcal{O}_X/p^n) \rightarrow H_{cdh}^d(X, W\mathcal{O}_X/p^n)$$

$$(5.2) \quad H_{Zar}^d(X, \tilde{C}_jW\mathcal{O}_X/p^n) \rightarrow H_{cdh}^d(X, \tilde{C}_jW\mathcal{O}_X/p^n)$$

are surjective.

*Proof.* We prove by induction on  $n \geq 1$ . For  $n = 1$ , there is a canonical surjective map of  $\mathbb{F}_p$ -algebras  $W\mathcal{O}/p \twoheadrightarrow \mathcal{O}$  on  $Sch/k$  such that the kernel  $\mathcal{I}$  is a sheaf of square zero ideals by [14, page 9]. Hence by [16, Lemma 12.1], the map  $H_{cdh}^i(X, W\mathcal{O}_X/p) \rightarrow H_{cdh}^i(X, \mathcal{O}_X)$  is an isomorphism for all  $i \geq 0$ . In particular, we get  $H_{cdh}^d(X, \mathcal{I}_X) = 0$ . Thus we get a commutative diagram of cohomology groups with exact rows

$$\begin{array}{ccccccc} H_{Zar}^d(X, \mathcal{I}_X) & \longrightarrow & H_{Zar}^d(X, W\mathcal{O}_X/p) & \longrightarrow & H_{Zar}^d(X, \mathcal{O}_X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & H_{cdh}^d(X, W\mathcal{O}_X/p) & \xrightarrow{\cong} & H_{cdh}^d(X, \mathcal{O}_X) & & \end{array}$$

The right vertical map is surjective by [4, Theorem 6.1]. We remark here that Theorem 6.1 of [4] is proved when the base field is of characteristic zero. However exactly the same argument works even if  $k$  is of positive characteristic as long as the resolution of singularities holds over  $k$ , which we have assumed throughout. Only extra ingredients needed are the results of [19] and the formal function theorem which are characteristic free. We refer to *loc. cit.* for details of the proof. We conclude from the above diagram that 5.1 holds for  $n = 1$ .

Now we assume that  $n \geq 2$  and the surjectivity holds in 5.1 for  $m \leq n - 1$ . We have the exact sequence of the sheaves of abelian groups on  $Sch/k$

$$0 \rightarrow W\mathcal{O}/p^{n-1} \rightarrow W\mathcal{O}/p^n \rightarrow W\mathcal{O}/p \rightarrow 0.$$



Considering the cohomology, we get the following commutative diagram with exact rows (since  $H^d$  is right exact on the Zariski or the  $cdh$ -site of  $X$ ).

$$\begin{array}{ccccccc} H_{Zar}^d(X, W\mathcal{O}_X/p^{n-1}) & \longrightarrow & H_{Zar}^d(X, W\mathcal{O}_X/p^n) & \longrightarrow & H_{Zar}^d(X, W\mathcal{O}_X/p) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_{cdh}^d(X, W\mathcal{O}_X/p^{n-1}) & \longrightarrow & H_{cdh}^d(X, W\mathcal{O}_X/p^n) & \longrightarrow & H_{cdh}^d(X, W\mathcal{O}_X/p) & \longrightarrow & 0 \end{array}$$

The right and the left vertical arrows are surjective by induction. Hence the middle vertical arrow is also surjective. This proves the surjectivity of the map in 5.1.

To prove the surjectivity of the map in 5.2, we apply exactly the same argument of induction as above and observe that for  $n = 1$ , the map  $\tilde{C}_j W\mathcal{O} \rightarrow \tilde{C}_j \mathcal{O}$  is surjective such that its kernel  $\tilde{C}_j \mathcal{I}$  is of the form  $\mathcal{I} \otimes_{\mathbb{F}_p} V$  for some  $\mathbb{F}_p$ -vector space  $V$ , and hence  $H_{cdh}^d(X, \tilde{C}_j \mathcal{I}) \cong H_{cdh}^d(X, \mathcal{I}) \otimes_{\mathbb{F}_p} V = 0$ . In particular, we get isomorphism  $H_{cdh}^d(X, \tilde{C}_j W\mathcal{O}_X/p) \xrightarrow{\cong} H_{cdh}^d(X, \tilde{C}_j \mathcal{O}_X)$ . Moreover, the map  $H_{Zar}^d(X, \tilde{C}_j \mathcal{O}_X) \rightarrow H_{cdh}^d(X, \tilde{C}_j \mathcal{O}_X)$  is surjective by [4, Proof of Theorem 6.2]. The proof now follows from the induction.  $\square$

**Proposition 5.2.** *Let  $X$  be a  $k$ -variety of dimension  $d$ . Then the natural maps*

$$\begin{aligned} & H_{Zar}^d(X, a_{Zar}\pi_{-1}(TC/p^n(-; p))) \rightarrow H_{cdh}^d(X, a_{cdh}\pi_{-1}(TC/p^n(-; p))); \\ & H_{Zar}^d\left(X, a_{Zar}\pi_{-1}\left(\tilde{C}_j TC/p^n(-; p)\right)\right) \rightarrow H_{cdh}^d\left(X, a_{cdh}\pi_{-1}\left(\tilde{C}_j TC/p^n(-; p)\right)\right) \end{aligned}$$

are surjective.

*Proof.* Since  $a_{\mathcal{C}}\pi_{-2}(TC(-; p)) = 0$  for  $\mathcal{C}$  being the Zariski or the  $cdh$ -site, the exact sequence

$$a_{\mathcal{C}}\pi_{-1}(TC(-; p)) \xrightarrow{p^n} a_{\mathcal{C}}\pi_{-1}(TC(-; p)) \rightarrow a_{\mathcal{C}}\pi_{-1}(TC/p^n(-; p)) \rightarrow a_{\mathcal{C}}\pi_{-2}(TC(-; p))$$

implies that  $a_{\mathcal{C}}\pi_{-1}(TC/p^n(-; p)) \cong a_{\mathcal{C}}\pi_{-1}(TC(-; p))/p^n$  and the same isomorphism holds for the  $\tilde{C}_j$ -sheaves. On the other hand, it is known (cf. [9]) that there are canonical isomorphisms

$$a_{\mathcal{C}}\pi_{-1}(TC(-; p)) \xrightarrow{\cong} \text{Coker}\left(a_{\mathcal{C}}W\mathcal{O}_X \xrightarrow{id-F} a_{\mathcal{C}}W\mathcal{O}_X\right)$$

and similar isomorphism holds for the  $\tilde{C}_j$ -sheaves, where

$$\tilde{C}_j W\mathcal{O}_X = \text{CoKer}(W\mathcal{O}_{X[T_1, \dots, T_j]} \rightarrow W\mathcal{O}_X).$$

Combining the above two isomorphisms, we conclude that there are canonical isomorphisms

$$(5.3) \quad \begin{aligned} a_{Zar}\pi_{-1}(TC/p^n(-;p)) &\xrightarrow{\cong} \text{Coker}\left(a_{Zar}W\mathcal{O}_X/p^n \xrightarrow{id-F} a_{Zar}W\mathcal{O}_X/p^n\right) \text{ and} \\ a_{cdh}\pi_{-1}(TC/p^n(-;p)) &\xrightarrow{\cong} \text{Coker}\left(a_{cdh}W\mathcal{O}_X/p^n \xrightarrow{id-F} a_{cdh}W\mathcal{O}_X/p^n\right). \end{aligned}$$

The similar isomorphisms hold for the  $\tilde{C}_j$ -sheaves. Since  $H^d$  is right exact on the Zariski or the  $cdh$ -site of  $X$ , it suffices to show that for every  $n \geq 1$ , the maps

$$(5.4) \quad H_{Zar}^d(X, W\mathcal{O}_X/p^n) \rightarrow H_{cdh}^d(X, W\mathcal{O}_X/p^n)$$

$$(5.5) \quad H_{Zar}^d(X, \tilde{C}_j W\mathcal{O}_X/p^n) \rightarrow H_{cdh}^d(X, \tilde{C}_j W\mathcal{O}_X/p^n)$$

are surjective. But this is proved in Lemma 5.1.  $\square$

**Theorem 5.3.** *Let  $X$  be a  $k$ -variety of dimension  $d$ . Then  $K_i(X, \mathbb{Z}/p^n) = 0 = \pi_i \tilde{C}_j K/p^n(X)$  for  $j \geq 0$  and  $i < -d - 1$ .*

*Proof.* We first show that

$$(5.6) \quad TC_i(X; p, \mathbb{Z}/p^n) = 0 = \pi_i \tilde{C}_j TC/p^n(-; p)(X) \text{ for } i < -d - 1.$$

The Zariski descent for  $TC/p^n(-; p)$  and  $\tilde{C}_j TC/p^n(-; p)$ , as shown in the proof of Corollary 3.2, gives a strongly convergent spectral sequence

$$E_2^{s,t} = H_{Zar}^s(X, a_{Zar}\pi_t(TC/p^n(-; p))) \Rightarrow TC_{t-s}(X; p, \mathbb{Z}/p^n)$$

and the similar spectral sequence holds for the homotopy groups of the  $\tilde{C}_j$  functors. Thus it suffices to show that  $TC_i(A; p, \mathbb{Z}/p^n) = 0$  for  $i < -1$  for any ring  $A$  which is essentially of finite type over  $k$ . But this has already been shown above. This proves 5.6.

The homotopy fibration sequence 2.1 gives the long exact sequence of homotopy groups

$$\cdots L_i^n(X) \rightarrow K_i(X, \mathbb{Z}/p^n) \rightarrow TC_i(X; p, \mathbb{Z}/p^n) \rightarrow L_{i-1}^n(X) \rightarrow \cdots$$

and one has a similar long exact sequence of the homotopy groups of the functors  $\tilde{C}_j$ . Lemma 4.1 and 5.6 together now imply that

$$K_i(X, \mathbb{Z}/p^n) = 0 = \pi_i \tilde{C}_j K/p^n(X) \text{ for } i < -d - 2$$

and there are exact sequences

$$TC_{-d-1}(X; p, \mathbb{Z}/p^n) \rightarrow L_{-d-2}^n(X) \rightarrow K_{-d-2}(X, \mathbb{Z}/p^n) \rightarrow 0,$$

$$\pi_{-d-1} \tilde{C}_j TC/p^n(-; p)(X) \rightarrow \pi_{-d-2} \tilde{C}_j L^n(X) \rightarrow \pi_{-d-2} \tilde{C}_j K/p^n(X) \rightarrow 0.$$

Thus we need to show that the first map in both the exact sequences are surjective.

We consider the following commutative diagram.

$$\begin{array}{ccccc}
 H_{Zar}^d(X, a_{Zar}\pi_{-1}(TC/p^n(-; p))) & \rightarrow & \mathbb{H}_{Zar}^{d+1}(X, TC/p^n(-; p)) & \leftarrow & TC_{-d-1}(X; p, \mathbb{Z}/p^n) \\
 \downarrow & & \downarrow & & \downarrow \\
 H_{cdh}^d(X, a_{cdh}\pi_{-1}(TC/p^n(-; p))) & \rightarrow & \mathbb{H}_{cdh}^{d+1}(X, TC/p^n(-; p)) & & \\
 \downarrow & & \downarrow & & \\
 H_{cdh}^d(X, a_{cdh}\pi_{-2}(L^n)) & \longrightarrow & \mathbb{H}_{cdh}^{d+2}(X, L^n) & \longleftarrow & L_{-d-2}^n(X)
 \end{array}$$

The left horizontal arrows of all the rows are isomorphisms by Lemmas 4.2 and 4.3. The right horizontal arrow of the top row is an isomorphism by the Zariski descent of  $TC$  (cf. [5], [1]). The right horizontal arrow in the bottom row is an isomorphism by Corollary 3.2. The lower vertical arrow on the left column is an isomorphism by 4.4. The upper vertical arrow on the left column is surjective by Proposition 5.2. A diagram chase shows that the long vertical arrow in the extreme right is surjective. The surjectivity of the map  $\pi_{-d-1}\tilde{C}_j TC/p^n(-; p)(X) \rightarrow \pi_{-d-2}\tilde{C}_j L^n(X)$  follows exactly in the same way using Lemmas 4.2, 4.3, Proposition 5.2 and Corollary 3.2.  $\square$

## 6. VANISHING AND HOMOTOPY INVARIANCE FOR RATIONAL $K$ -THEORY

For any presheaf of spectra  $\mathcal{E}$  on  $Sch/k$ , let  $\mathcal{E}_{\mathbb{Q}}$  denote the direct colimit over the multiplication maps  $\mathcal{E} \xrightarrow{n} \mathcal{E}$  by positive integers (cf [18]). Then  $\mathcal{E}_{\mathbb{Q}}$  is a presheaf of spectra on  $Sch/k$  such that  $\pi_i(\mathcal{E}_{\mathbb{Q}}) \cong \pi_i(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$  for  $i \in \mathbb{Z}$ . Our goal now is prove the vanishing of the rational  $K$ -theory and  $K_{\mathbb{Q}}$ -regularity in degrees below minus the dimension of a  $k$ -variety, where  $k$  is an infinite perfect field of positive characteristic as before. Let  $\mathcal{K}_{\mathbb{Q}}$  and  $\mathcal{HC}_{\mathbb{Q}}$  denote the presheaves of rational  $K$ -theory and rational cyclic homology spectra on  $Sch/k$ . Let  $\widetilde{\mathcal{HN}}_{\mathbb{Q}}$ ,  $\widetilde{\mathcal{HP}}_{\mathbb{Q}}$  and  $\widetilde{\mathcal{HC}}_{\mathbb{Q}}$  denote the presheaves of spectra on  $Sch/k$  given by  $U \mapsto HN(U \otimes \mathbb{Q})$ ,  $U \mapsto HP(U \otimes \mathbb{Q})$  and  $U \mapsto HC(U \otimes \mathbb{Q})$ , where  $U$  is considered as a scheme over  $\mathbb{Z}$ , and  $HN$ ,  $HP$  and  $HC$  respectively are the presheaves of negative cyclic homology, periodic cyclic homology and cyclic homology spectra on the category of schemes over  $\mathbb{Z}$ . There is a homotopy fibration sequence of the Eilenberg-Mac Lane spectra (cf. [13, Section 5.1])

$$(6.1) \quad \widetilde{\mathcal{HN}}_{\mathbb{Q}} \rightarrow \widetilde{\mathcal{HP}}_{\mathbb{Q}} \rightarrow \Omega^{-2}\widetilde{\mathcal{HC}}_{\mathbb{Q}}.$$

There is a generalized Chern character map (cf. [13, Section 8.4])

$$\mathcal{K}_{\mathbb{Q}} \xrightarrow{ch} \widetilde{\mathcal{HN}}_{\mathbb{Q}}.$$

Let  $\mathcal{K}_{\mathbb{Q}}^{\text{inf}}$  denote the homotopy fiber of the above map of spectra.

**Lemma 6.1.** *The presheaf of spectra  $\mathcal{K}_{\mathbb{Q}}$  on  $Sch/k$  satisfies  $cdh$ -descent.*

*Proof.* Since our schemes are defined over  $k$  which is of positive characteristic, we see that the presheaf of spectra  $\widetilde{\mathcal{HN}}_{\mathbb{Q}}$  is contractible. Hence using the homotopy fibration sequence

$$\mathcal{K}_{\mathbb{Q}}^{\text{inf}} \rightarrow \mathcal{K}_{\mathbb{Q}} \xrightarrow{ch} \widetilde{\mathcal{HN}}_{\mathbb{Q}},$$

it suffices to show that the presheaf of spectra  $\mathcal{K}_{\mathbb{Q}}^{\text{inf}}$  satisfies  $cdh$ -descent on  $Sch/k$ . To this end, it suffices to show that  $\mathcal{K}_{\mathbb{Q}}^{\text{inf}}$  satisfies all the conditions of Theorem 3.1. It satisfies excision by [3, Theorem 01] and it is invariant under infinitesimal extension by [7, Main Theorem].  $\mathcal{K}_{\mathbb{Q}}^{\text{inf}}$  satisfies Nisnevich descent by [18, Theorem 10.8] and it satisfies the Mayer-Vietoris property for blow-up under regular closed embeddings by [19, Theorem 2.1]. This completes the proof of the lemma.  $\square$

**Corollary 6.2.** *Let  $X$  be a  $k$ -variety of dimension  $d$ . Then one has  $K_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 = \pi_i(\widetilde{C}_j \mathcal{K}_{\mathbb{Q}})(X)$  for  $j \geq 0$  and  $i < -d$ .*

*Proof.* This follows directly from Lemma 6.1, the spectral sequence of Corollary 3.3 and from [16, Theorem 12.5] since  $a_{cdh} \pi_i(\mathcal{K}_{\mathbb{Q}}) = 0$  for  $i < 0$ .  $\square$

**Lemma 6.3.** *Let  $\mathcal{KH}$  denote the presheaf of homotopy invariant  $K$ -theory on  $Sch/k$  (cf. [24]). Then  $\mathcal{KH}$  satisfies  $cdh$ -descent on  $Sch/k$ .*

*Proof.* The proof of the lemma follows exactly in the same way as the case when  $k$  is of characteristic zero (cf. [8, Theorem 6.4]). Only thing one needs is that the resolution of singularities should hold over  $k$ . We skip the details.  $\square$

**Corollary 6.4.** *Let  $X$  be a  $k$ -variety of dimension  $d$  and let  $n$  be a positive integer prime to  $p$ . Then  $K_i(X, \mathbb{Z}/n) = 0 = \widetilde{C}_j K_i(X, \mathbb{Z}/n)$  for  $j \geq 0$  and  $i < -d$ .*

*Proof.* The statement of the corollary for  $KH_i(X)$  and  $\widetilde{C}_j KH_i(X)$  follows from Lemma 6.3 and Corollary 3.3. For  $n$  prime to  $p$ , the natural map  $K_i(X, \mathbb{Z}/n) \rightarrow KH_i(X, \mathbb{Z}/n)$  is an isomorphism by [24, Proposition 1.6], and the same conclusion holds for  $\widetilde{C}_j K_i(X)$ . Now the corollary follows from the exact sequence

$$0 \rightarrow KH_i(X) \otimes \mathbb{Z}/n \rightarrow KH_i(X, \mathbb{Z}/n) \rightarrow \text{Tor}(KH_{i-1}(X), \mathbb{Z}/n) \rightarrow 0.$$

$\square$

**Proof of Theorem 1.3:** For any abelian group  $A$ , let  ${}_n A$  denote the subgroup of  $n$ -torsion elements of  $A$ . It follows from Corollary 6.2 that  $K_i(X)$  and  $\widetilde{C}_j K_i(X)$  are torsion groups for  $i < -d$ . Thus we only need to show that these groups have no torsion whenever  $i < -d - 2$ . For any positive integer  $n$ , there is a short exact sequence

$$0 \rightarrow K_i(X) \otimes \mathbb{Z}/n \rightarrow K_i(X, \mathbb{Z}/n) \rightarrow_n K_{i-1}(X) \rightarrow 0.$$

Theorem 5.3 and Corollary 6.4 now immediately imply that  $K_i(X)$  and  $\tilde{C}_j K_i(X)$  are divisible groups for  $i < -d - 1$ . Moreover, these groups are torsion-free for  $i < -d - 2$ . In particular,  $K_i(X) = 0$  and  $X$  is  $K_i$ -regular for  $i < -d - 2$ .  $\square$

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School of Mathematics  
Tata Institute Of Fundamental Research  
Homi Bhabha Road  
Mumbai, 400005, India  
email : amal@math.tifr.res.in